Surface Bodies in Dimension 2

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We examine the volume difference between a convex body K and it's surface body K_s in dimension 2. First, some background and definitions are given. Then we look at several examples leading up to a surprisingly simple formula for the volume difference in a polygon in dimension 2. We find that in the polygonal case $\lim_{s\to 0} \mathbf{Vol}_n(K/K_s) = \sum_i \sin(\theta_i)s^2/6$ where θ_i are the angles made at the vertices of the polygon. Moreover, it appears that all convex bodies in $\mathbb{R}^2$ have $\lim_{s\to 0} \mathbf{Vol}_n(K/K_s)$ is of order s^2. This is in contrast to the floating body which may admit a variety of dependencies based on the shape of the body.

I. INTRODUCTION

The surface body is defined to be the intersection of all half-spaces H of hyperplanes which cut off a distance s from the boundary of a given convex shape K. In this paper we will examine the behavior of the volume difference between several convex bodies and their surface bodies and compute K_s. Then we generalize the techniques used in II to find a formula for the volume difference in an arbitrary polygon.

II. EXAMPLES

A. Circle

let

$$B = \{x \in \mathbb{R}^2 : ||x||_2 \le r\}. \tag{1}$$

By symmetry B_s must be a ball which we will denote as having radius r_s. If we have any hyperplane cutting off an arc of length s then we can find all the other ones cutting or an arc length of s by simply shifting the hyperplane along the boundary of the circle. Thus, we know that the surface body itself must be a circle since all of these hyperplanes have the same smallest distance from the hyperplane to the center of the circle. An easy calculation shows that

$$r_s = r\cos(s/2r). \tag{2}$$

Now we look at the volume difference

$$\mathbf{Vol}_2(B) - \mathbf{Vol}_2(B_s) = \pi r^2 \sin^2((s/2r)) \approx \frac{\pi}{4}s^2. \tag{3}$$

It is rather surprising to note that the volume difference is independent of r. In the appendix we show that in all other dimensions the volume difference for the ball is dependent on r.

B. Square

let

$$S = \{x \in \mathbb{R}^2 : ||x||_\infty \le \ell\}. \tag{4}$$

For s small, it is sufficient to consider one vertex of the square and by symmetry, we compute the volume difference. We shift the square so that the vertex under consideration lies at the origin and the square lies in quadrant I. If $s < \ell/2$ then we know that the surface body will touch the side of the square at a distance s from the origin. This is because s is as far as a hyperplane cutting off a distance s from the boundary may go from the origin.

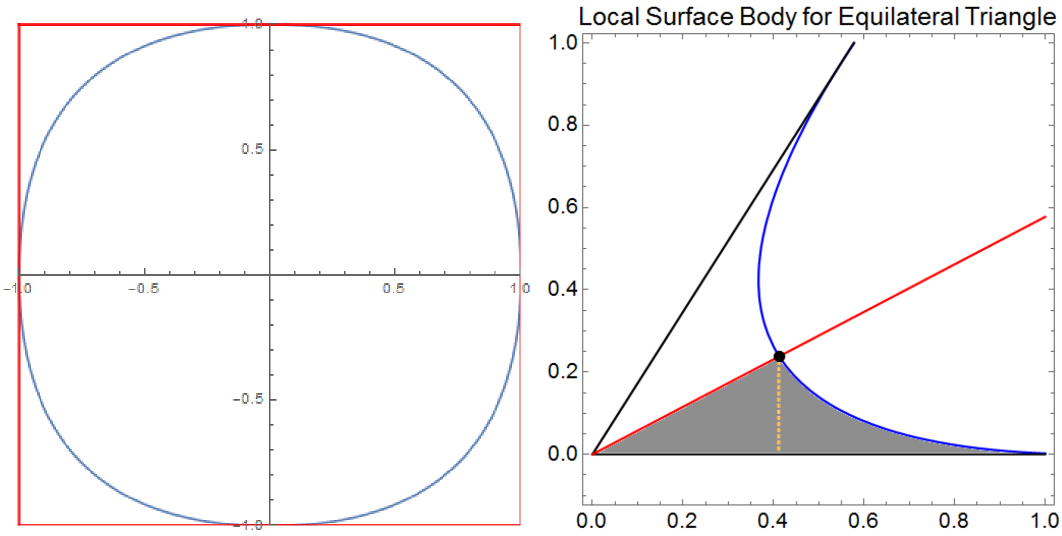

Figure 1: Surface bodies for the square and the equilateral triangle. Note that the surface body for the triangle is locally symmetric about the line $y = \tan(\pi/6)$ and thus the area difference can be evaluated by integrating in 2 separate parts

A hyperplane that cuts off surface area s is a line, $y = mx + b$, that passes through points $P_1 = (x_1, 0)$ and $P_2 = (0, y_2)$ which satisfy $s = x_1 + y_2 \implies x_1 = s - y_2$. Thus we get

$$b = y_2 \text{ and } m = \frac{y_2}{y_2 - s}. \tag{5}$$

Plug these in to obtain thee result for a general hyperplane cutting off a set of length s.

$$y = \frac{y_2}{y_2 - s} x + y_2. \tag{6}$$

Thus we have a family of curves $f(x, y, y_2) = y_2 x + y_2(y_2 - s) - y(y_2 - s)$. Now we compute the envelope by setting $f = 0$ and $\frac{\partial f}{\partial y_2} = 0$:

$$\frac{\partial f}{\partial y_2} = x + 2y_2 - s - y = 0 \implies y_2 = \frac{s + y - x}{2}. \tag{7}$$

Substituting this back into f will give

$$(x - y + s)^2 - 4sx = 0. \tag{8}$$

Note that we will only examine this curve within the square defined by $x, y \in [0, s]$.

Now to compute the volume difference, we compute the following integral and multiply by 4:

$$\int_0^s x + s - 2\sqrt{sx}dx = \frac{x^2}{2} + sx - \frac{4}{3s}(sx)^{3/2}|_0^s = \frac{s^2}{6} \tag{9}$$

Thus for sufficiently small s we have that

$$\mathbf{Vol}_2(S) - \mathbf{Vol}_2(S_s) = \frac{2s^2}{3} \tag{10}$$

Again, we find that the result is scale invariant.

C. Equilateral Triangle

Define the equilateral triangle T to be of side length t with it's center at the origin. Similar to the square we can look at the individual corners of the triangle to find the curve which represents the surface body at each vertex for sufficiently small s. If we place one vertex of the triangle at the origin and one side along the x axis we get

$$s = -\frac{b}{m} + \sqrt{\left(\frac{b}{\sqrt{3}-m}\right)^2 + \left(\frac{b\sqrt{3}}{\sqrt{3}-m}\right)^2} \tag{11}$$

where m and b are the slope and of and x intercept of the hyperplane cutting off a set of length s from the triangle. This can be simplified to

$$s = b\left(-\frac{1}{m} + \frac{2}{\sqrt{3}-m}\right) \tag{12}$$

And thus, the equation which expresses all of the hyperplanes can be parameterized by m and is given by

$$y = mx + s\left(\frac{m(\sqrt{3}-m)}{3m-\sqrt{3}}\right). \tag{13}$$

Next we calculate the envelope of this curve by making it into a 2nd order polynomial in m and set the discriminant equal to 0.

$$m^2(s-3x) + m\left(\sqrt{3}x + 3y - \sqrt{3}s\right) - \sqrt{3}y = 0 \tag{14}$$

$$\sqrt{3}\left(x + \sqrt{3}y - s\right)^2 + 4y(s-3x) = 0 \tag{15}$$

and we have the curve which locally represents the surface body. Solving for y we obtain that

$$y = \frac{s + 3x \pm 2\sqrt{-2s(s-3x)}}{3\sqrt{3}} \tag{16}$$

where we only consider $x > \frac{2s}{3}$. Next we evaluate the volume difference between the surface body and the triangle. We observe that the volume is symmetric across the line $y = x\tan 15^\circ$ $(y = \frac{x}{\sqrt{3}})$. Thus, if we find the intersection of this line with (16) then we can split the total volume difference into an area given by the boundary of the surface body and the volume under the curve $y = \frac{x}{\sqrt{3}}$. We find the intersection by solving for x and we get that

$$x = \frac{3s}{8}. \tag{17}$$

We know that the other bound of the integral will simply be s because $(s,\, 0)$ is the furthest point from the origin from which we may have a hyperplane which cuts off a set of length s from ∂T(the shaded area in 1). So we evaluate the integral using (16) with the negative sign.

$$\int_{\frac{3s}{8}}^{s} \frac{s + 3x - 2\sqrt{-2s(s-3x)}}{3\sqrt{3}}\,\mathrm{d}x = \frac{7s^2}{128\sqrt{3}}. \tag{18}$$

The other integral is bounded by the line which split the corner

$$\int_0^{\frac{3s}{8}} \frac{x}{\sqrt{3}}\,\mathrm{d}x = \frac{9}{128}\sqrt{3}s^2 \tag{19}$$

Then we multiply by to to obtain

$$\mathbf{Vol}_2(T) - \mathbf{Vol}_2(T_s) = \frac{1}{96}\left(27 - 10\sqrt{3}\right)s^2 \tag{20}$$

D. Almost Polygonal Bodies

Here we consider a construction for almost polygonal bodies given in [1]. We define a sequence $(a_n)_{n\in\mathbb{N}}$ take the convex hull of points of the form $(\pm a_n,\ a_n^2)$ and $(\pm a_n,\ 2-a_n^2)$ where $(a_n)_{n\in\mathbb{N}}$ meets some simple requirements for converging to 0 (see 2 for picture). By the symmetries of the object then we can analyze the surface body using only one side of it.

Here we consider sequences of positive real numbers such that

$$1 = a_0 \geq ...a_n \geq a_{n+1} \geq ... \geq 0 \ \ \forall n \in \mathbb{N} \tag{21}$$

$$a_{n-1} - a_n \geq a_n - a_{n+1} \ \ \forall n \in \mathbb{N} \tag{22}$$

$$\lim_{n\to\infty} a_n = 0. \tag{23}$$

However before we look at the behavior we must define when the surface body starts to notice that the shape is getting smooth. Take the distance between any 2 points along the edge of the body

$$s = \sqrt{(a_{n-1}-a_n)^2 + (a_{n-1}^2 - a_n^2)^2} + \sqrt{(a_n - a_{n+1})^2 + (a_n^2 - a_{n+1}^2)^2} \tag{24}$$

and we can simplify it to

$$s = (a_{n-1}-a_n)\sqrt{1+(a_{n-1}+a_n)^2} + (a_n - a_{n+1})\sqrt{1+(a_n+a_{n+1})^2} \tag{25}$$

which suggests the following definition.

Definition II.1 *Define n_s to be the first n such that*

$$a_{n_s} - a_{n_s+1} < s/4. \tag{26}$$

Proposition II.2 *Let a_n be a sequence satisfying (21), (22), and (23) and let K(a) be the body discussed in I. Then $\exists$ a, c, and s_0 such that we have for all $0 \leq s \leq s_0$,*

$$\frac{1}{c}s^2 + \frac{1}{c}s^2 k_s \leq \textbf{\textit{Vol}}_2(K/K_s) \leq cs^2 + cs^2 k_s \tag{27}$$

From [1] we have that the smooth part of the body will behave like

$$\lim_{s\to 0} \frac{\textbf{Vol}_2(K/K_s)}{s^2} = C \tag{28}$$

where C is simply a constant. So we must look at the parts of the body which are polygonal and find an estimate of the volume differences. To do this we observe that as $s \to 0$ it will become less than any side of the boundary and thus we only consider the corners of the body. We define $a = s/2$ to simplify some of the algebra. We may assume that any hyperplane (line in this case) will cut off between a and $2a$ from either side of the corner. We will do the calculation for the a case and the $2a$ case follows quickly from it.

The cut off area is a triangle with area given by

$$A_a = \frac{1}{2} b_a h_a \tag{29}$$

where b_a is the base of the triangle given by the length of the intersection between the hyperplane and the body and h_a is the height of the triangle or the minimum distance hyperplane to the corner.

By the law of cosines and the Pythagorean theorem we get

$$b_a^2 = a^2 + a^2 - 2(a)(a)\cos\theta = 2a^2(1-\cos\theta) \tag{30}$$

$$h_a^2 + (\frac{1}{2}b_a)^2 = a^2 \tag{31}$$

where (31) comes from the fact that the triangle is isosceles. With a small amount of algebra we get from (31) that

$$h_a^2 = \frac{1}{2}a^2(1+\cos\theta). \tag{32}$$

By combining this with (29) we obtain

$$A_a^2 = \frac{1}{4}a^4\sin^2\theta \tag{33}$$

and similarly for the $2a$ case we obtain

$$A_{2a}^2 = 4a^4\sin^2\theta. \tag{34}$$

So then we have that

$$\frac{1}{8}s^2\sin\theta < A < \frac{1}{4}s^2\sin\theta \tag{35}$$

are the proper bounds on A for each of the corners. Next we must add all of the triangles together to get the full area given by the formula

$$\sum_{\text{poly}} A < \frac{1}{4}s^2\sum_{i=0}^{k_s}\sin\theta_i < \frac{1}{4}s^2 k_s\sin\theta_0 \tag{36}$$

where θ_i are the angles for each of the respective corners. The second inequality comes from the fact that $\pi/2 < \theta_i < \pi \;\; \forall i$ and the sequence of θ_i's is strictly increasing. Thus we obtain the simple relation that

$$\sum_{\text{poly}} A \sim s^2 k_s. \tag{37}$$

Since the total end behavior of the volume differences will be the tighter of the two bounds provided we have that

$$\text{Vol}_2(V/V_s) \sim s^2 + s^2 k_s. \tag{38}$$

But we know that K_s will only increase with s so we get that the smooth portion aways dominates in the small s limit.

E. Comments on Examples

It is clear to see that all of these bodies admit s^2 behavior in which the coefficient is not dependent on scaling. We will see in the next section that in fact this does hold true for all polygons. However this remains to be shown for all convex bodies in $\mathbb{R}^2$.

The coefficients themselves also seem to be somehow superficially related to the body. Take, for instance, the case of the circle which admits a factor of π which does not appear in any other of the bodies we have examined. It may be possible to guess some factors in the coefficient simply by inspection.

III. ARBITRARY POLYGON

In this section we consider the case of the polygon with an enumeration of the angles given indexed as θ_i. Since we know that each side is finite we know that s will become smaller than any side and the surface body will touch the boundary between every vertex. Thus, we may reduce the case of an arbitrary polygon to finding the surface body around each vertex.

Thus we consider a vertex at the origin with an arbitrary angle θ made between 2 lines one of which lies on the x axis. We quickly find that out hyperplane will intersect the angle at the points

$$\left(-\frac{b}{m}, 0\right) \text{ and } \left(\frac{b}{\tan(\theta)-m}, \frac{b\tan(\theta)}{\tan(\theta)-m}\right). \tag{39}$$

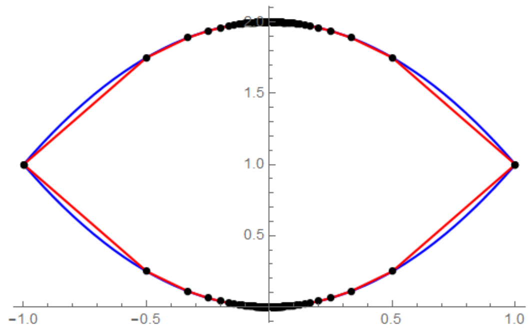

Figure 2: A diagram of the almost polygonal body given in II D. The body itself is given by the lines inside the outter shape. note the reflective symmetries of the object and the accumulation points at the origin and (0, 2).

When we calculate the surface area cut off by this hyperplane we get that

$$s = \frac{b}{|\sin(\theta) - m\cos(\theta)|} - \frac{b}{m}. \tag{40}$$

Next we solve for b and substitute back into the original formula for the hyperplane to get

$$y = mx + \frac{s}{\frac{1}{|\sin(\theta)-m\cos(\theta)|} - \frac{1}{m}}. \tag{41}$$

Because of the absolute value, now is an opportune time to split our problem into evaluating the acute and obtuse cases.

For the acute case we know that $\min\{\cos(\theta), \sin(\theta)\} \geq 0 \geq m$ so we have that the quantity $\sin(\theta) - m\cos(\theta)$ is non-negative. Thus we have that

$$m^2(x\cos(\theta) + x - s\cos(\theta)) + m(s\sin(\theta) - x\sin(\theta) - y\cos(\theta) - y) + y\sin(\theta) = 0 \tag{42}$$

Which we can easily calculate the envelope to be

$$(s\sin(\theta) - x\sin(\theta) - y\cos(\theta) - y)^2 - 4(x\cos(\theta) + x - s\cos(\theta))y\sin(\theta) = 0 \tag{43}$$

Solving for y we obtain

$$y = \frac{\sin(\theta)\left(\cos(\theta)(x-s) \pm 2\sqrt{s(\cos(\theta)(x-s)+x)} + s + x\right)}{(\cos(\theta)+1)^2}. \tag{44}$$

We will evaluate the acute case in much the same way that we did the triangle. We note that the body will be symmetric about the line $y = \tan(\theta/2)$ and calculate the intersection. If we do this we obtain that the intersection point is

$$x = \frac{s}{4}(\cos(\theta) + 1). \tag{45}$$

Because we know that the derivative of the surface body cannot be positive when our splitting line intersects it, we know that only the negative solution to (44) will contribute to the integral. Again, we have that the other bound of our integral will be s since it is the farthest point the hyperplane can reach to. Thus we must evaluate the integral

$$\int_{\frac{s}{4}(\cos(\theta)+1)}^{s} \frac{\sin(\theta)\left(\cos(\theta)(x-s) - 2\sqrt{s(\cos(\theta)(x-s)+x)} + s + x\right)}{(\cos(\theta)+1)^2}\,\mathrm{d}x = \frac{s^2}{96}\sin(\theta)(5 - 3\cos(\theta)). \tag{46}$$

The other integral is considerably easier to evaluate and is given by

$$\int_0^{\frac{s}{4}(\cos(\theta)+1)} x\tan(\theta/2)\,\mathrm{d}x = \frac{s^2}{32}(\cos(\theta)+1)^2\tan\left(\frac{\theta}{2}\right). \tag{47}$$

Finally, we obtain that for an acute angle the difference in area is

$$\frac{\sin(\theta)}{6}s^2. \tag{48}$$

A remarkably simple result.

Next we move on to the obtuse case. Here we cannot make any assumptions about the sign of the term inside the absolute value in (40). To find an answer we look at the domain of m which is relevant to the surface body. Since we are considering the obtuse angles only we get the condition

$$0 < m < \tan(\theta) \implies 0 < \sin(\theta) - m\cos(\theta) \tag{49}$$

So we can simply take the absolute value signs away as in the acute case. We may then repeat the same argument as above until we reach the integrals. We split up our integral into a piece in the first quadrant and another in the second quadrant. The bounds for the second quadrant are simple if we recall that the surface body can only go a length s on the line $y = \tan(\theta)$. Thus we get the bound

$$x = \frac{s}{\sec(\theta)}. \tag{50}$$

The integral for the second quadrant will be the the difference between the curve of the surface body and $y = \tan(\theta)$. Putting it all together we get

$$\int_{\frac{s}{\sec(\theta)}}^{0} \frac{\sin(\theta)\left(\cos(\theta)(x-s) - 2\sqrt{s(\cos(\theta)(x-s)+x)} + s + x\right)}{(\cos(\theta)+1)^2} - x\tan(\theta)\,\mathrm{d}x. \tag{51}$$

which has value

$$\frac{s^2\sin(\theta)\cos(\theta)\left(12\cos(\theta) + \cos(2\theta) + 16\sqrt{-\cos(\theta)} - 5\right)}{12(\cos(\theta)+1)^3} \tag{52}$$

The right side is relatively simple to set up and we get that it's value is

$$-\frac{s^2\sin(\theta)\left(4\left(4\sqrt{-\cos(\theta)} - 3\right)\cos(\theta) + 3\cos(2\theta) + 1\right)}{12(\cos(\theta)+1)^3} \tag{53}$$

which admits no easy simplification. However, when these two pieces are added together we get that the total volume difference simplifies to

$$\frac{\sin(\theta)}{6}s^2. \tag{54}$$

Thus, we have found that in all cases for sufficiently small s, the volume difference for any polygon will follow the simple formula

$$\mathbf{Vol}_2(K) - \mathbf{Vol}_2(K_s) = \frac{s^2}{6}\sum_{\text{poly}} \sin(\theta_i) \tag{55}$$

IV. SUMMARY

It is clear that 2 dimensional shapes admit a very narrow set of behaviors as in the small s limit.

Acknowledgments

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Appendix: COEFFICIENT BEHAVIOR OF ALMOST POLYGONAL BODY

Now that we have a general formula for the surface body of an angle we may evaluate the coefficient of the surface body of the almost polygonal body described in II D. We can provide a lower bound to the difference between the surface bodies by

$$4 * \frac{s^2}{6} \sum_{i=0}^{k_s} \sin(\theta_i) < \mathbf{Vol}_2(K/K_s). \tag{A.1}$$

However, it is easy to see geometrically that eventually this expression will approach the volume difference in the limit. To evaluate the coefficient we must find a relation between the values of the sequence and the sine of their angle. A quick application of the formula for a cross product given an angle yeilds

$$\sin(\theta_i) = \frac{a_{i-1} - a_{i+1}}{\sqrt{(1 + (a_{i-1} + a_i)^2)(1 + (a_i + a_{i+1})^2)}} \tag{A.2}$$

So then we can substitute it back in to get that

$$\mathbf{Vol}_2(K/K_s) = \frac{2s^2}{3} \sum_{i=1}^{\infty} \frac{a_{i-1} - a_{i+1}}{\sqrt{(1 + (a_{i-1} + a_i)^2)(1 + (a_i + a_{i+1})^2)}} \tag{A.3}$$

should approximate the volume difference as $s \to 0$. From that it is an exercise in calculus to obtain the bounds

$$\frac{1}{5}\mathcal{C}(a_1) < \lim_{s \to 0} \frac{\mathbf{Vol}_2(K/K_s)}{s^2} < \mathcal{C}(a_1) \tag{A.4}$$

where

$$\mathcal{C}(a_1) = \frac{2(1 + a_1)}{3} \tag{A.5}$$

using the fact that $0 < a_i < 1 \;\; \forall i$. So then our possible outcomes for the coefficient are actually severely limited. If we relax our bounds a little bit we can obtain the even simpler formula that works for every sequence that we may consider

$$\frac{2s^2}{15} < \mathbf{Vol}_2(K/K_s) < \frac{4s^2}{3}. \tag{A.6}$$

Appendix: VOLUME DIFFERENCE BEHAVIOR OF $\mathbf{B}_n^2$

Let rB_n^2 be the euclidean ball in dimension n with radius r. We begin with the formula given in [1] using the curvature of a smooth, convex body K in $\mathbb{R}^n$.

$$d_n \lim_{s \to 0} \frac{\mathbf{Vol}_n(rB_2^n/(rB_2^n)_s)}{s^{\frac{2}{n-1}}} = \int_{\partial(rB_2^n)} \frac{\kappa^{\frac{1}{n-1}}}{f^{\frac{2}{n-1}}} \, \mathrm{d}\mu_{\partial(rB_2^n)} \tag{A.1}$$

where $f = 1$(see [2]), d_n is a constant which depends only on the dimension n, and κ is the Gaussian curvature given as a function of the boundary. But for a circle we have that $\kappa = 1/r$. Thus we can simplify (A.1) to

$$d_n \lim_{s \to 0} \frac{\mathbf{Vol}_n(rB_2^n/(rB_2^n)_s)}{s^{\frac{2}{n-1}}} = \frac{n|B_2^n|}{r^{\frac{n(n-2)}{n-1}}}. \tag{A.2}$$

Next we note that by symmetry of the sphere we get that the surface body must also be a ball in dimension n with radius r_s

$$|rB_2^n| - |(rB_2^n)_s| = |B_2^n|(r^n - r_s^n). \tag{A.3}$$

Putting these together we obtain

$$d_n \lim_{s \to 0} \frac{(r^n - r_s^n)}{s^{\frac{2}{n-1}}} = \frac{n}{r^{\frac{n(n-2)}{n-1}}}. \tag{A.4}$$

Which implies that for $s << r$ we get

$$r_s \sim \sqrt[n]{r^n - \frac{n}{d_n}\left(\frac{s^2}{r^{n(n-2)}}\right)^{\frac{1}{n-1}}}. \tag{A.5}$$

[1] Schütt, C., & Werner, E. (2003). Surface bodies and p-affine surface area. Elsevier Inc. doi: https://case.edu/artsci/math/werner/publications/surface

[2] f is defined in [1] to be a probability density along the boundary but we are only considering cases with a uniform boundary so we need not worry about it.