A Review of The 2-Dimensional Quantum Hall Effect

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The 2-dimensional quantum Hall effect is one of the most surprising consequences of quantum mechanics. This experiment measured the Hall and the longitudinal voltages across a AsGe semiconductor in an attempt to quantify e^2/h. Although a sensible looking graph of the Hall voltage was obtained, it was found that the measured values for e^2/h were off by a scale factor of roughly 8.56. The generally agreed upon value for e^2/h is 25818Ω and we obtained 15050 ± 38 Ω. Several suggestions as to why this is are put fourth including a wrong value for the resistor or a misread lockin. When the obtained values for the Hall resistance were fitted to different values of e^2/h, no regular pattern was found in the levels observed.

I. INTRODUCTION

The 2-dimensional quantum Hall effect was first predicted in the 1975 by Ando, Matsumoto, and Uemura and verified in 1980 by Klaus von Klitzing[1]. Since then it has been repeated in multiple experiments and is often used a simple model for the more complicated fractional quantum Hall effect.

In this experiment, currents were run across a semiconductor in an attempt to measure the 2-dimensional quantum Hall effect in which the Hall resistivity of a material is quantized due to energy levels rising above the Fermi energy. The energy levels of the material are dependent on an external magnetic field which is gradually increased in order to make the allowed energy states of the charge carriers move above the Fermi energy. Once the states are above the Fermi energy, they can no longer hold charge carriers in the conduction band, increasing the overall resistivity of the material in discrete levels based on which bands are below the Fermi energy.

II. THEORY

A. The Classical Hall effect

All of the following subsection is a part of or can easily be derived from [2].

In order for one to understand the 2-dimensional quantum Hall effect, it is useful to have an understanding of the classical Hall effect. In the classical Hall effect, a current is run over a conductor in an applied magnetic field. The magnetic field induces cyclotron motion in the charge carriers, causing them to congregate on one side of the sample as seen in Fig. 1.

The circular motion does not effect the movements of the charge carriers significantly in the middle of the conductor. However, on the edges (the left edge in the case of Fig. 1) the charge carriers cannot complete a revolution due to the boundary conditions and they simply stay on the side of the sample. This congregation of charge carriers changes the overall magnetic field and thus creates a new voltage across the crystal V_H, the Hall voltage.

Working this out precisely is relatively simple. First, we use the Lorentz force law to conclude that the force felt by the charges is given by

$$F_w = ev_d B_z \tag{1}$$

where e is the charge of the electron, v_d is the drift velocity, B_z is the Applied electric field, and F_w is the force on the electrons across the width of the sample. Next we use the fact that $F = d \times \rho$ where ρ is some constant potential to say that

$$V_H = \frac{F_w e}{w} \tag{2}$$

where w is the width of the sample. Next we use the current across the sample defined in terms of the drift velocity

$$I = n \cdot e \cdot t \cdot w \cdot v_d \tag{3}$$

where n is the number density of electrons and t is the height of the sample to obtain

$$V_H = I\frac{B_z}{nte}. \tag{4}$$

This looks very much like Ohm's law so we choose to define

$$R_H = \frac{V_H}{I} = \frac{1}{ne}. \tag{5}$$

This is the Hall resistance. It is defined in this way for both the classical and quantum Hall effects.

B. The Quantum Hall Effect

The mathematics in the following section can all be found in [1].

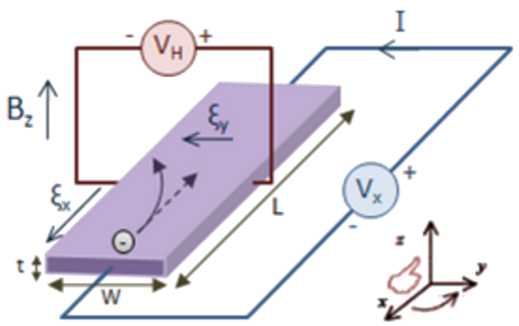

Figure 1: A diagram of the setup for a Hall effect (both classical and quantum). A current is run across a conductor and the charge carriers are deflected by the magnetic field to one side of the sample, causing a overall voltage difference across it. Diagram courtesy of `https://en.wikipedia.org/wiki/Hall_effect`.

The 2-dimensional quantum Hall Effect appears much the same as the classical Hall effect where the cyclotron motion of the charge carriers causes a change in voltage across the material.

We consider a 2-dimensional sheet of charge carriers at low temperature where x is the direction of a current across the sheet an y is the direction perpendicular to x in the plane of the sheet. To this we apply the modified 2-dimensional Schrdinger equation for an applied magnetic field given by

$$\left[E_s + \frac{(i\hbar + e\mathbf{A})^2}{2m_e} + U(y)\right]\Psi(x,y) = E\Psi(x,y) \quad (6)$$

where E_s is the spin energy, $\mathbf{A}$ is the magnetic potential, m_e is the mass of the electron, and $U(y)$ is the potential confining the particles to the sheet which is only dependent on y since we wish for current to move freely in the x direction. We define our confining potential as the parabolic equation

$$U(y) = \frac{1}{2}m_e\omega_0 y^2. \quad (7)$$

Clearly, this potential is not exact, however it is solvable and produces the 2-dimensional quantum Hall effect and is therefore useful for understanding. Note that the motion in the x direction is that of a free particle and so the equation can be separated into

$$\Psi = \phi(x)\chi(y) \quad (8)$$

where $\phi(x)$ is given by

$$\phi = \frac{1}{\sqrt{L}}e^{i\mathbf{k}_x\cdot\mathbf{x}} \quad (9)$$

where $\mathbf{k}$ is the momentum of the particle in the x direction and $\mathbf{x}$ is the position in the x direction. If this is plugged into 6 we obtain

$$\left[E_s + \frac{(i\hbar + e\mathbf{B}y)^2}{2m_e} + \frac{p_y^2}{3M_e} + \frac{1}{2}m_e\omega_0 y^2\right]\chi(x,y) = E\chi(x,y) \quad (10)$$

where B is the magnetic field. Now we may write down the eigenvalues in this system as

$$E = E_s + (n + \frac{1}{2})\hbar\omega + \frac{\hbar^2k^2}{2m_e}\frac{\omega_0^2}{\omega^2} \quad (11)$$

where $\omega = \omega_0^2 + |e|\mathbf{B}/m_e$, e is the charge of the electron and $n \in \mathbb{N}$ are the allowed energy levels of the system. It is clear to see that the energy levels of the system are discrete so we can say that there is finite amount of space for the electrons in the sample. As the magnetic field increases, the allowed energy levels increase. Because we are dealing with fermions at a low temperature, we know that they can only take on states which are below the Fermi energy. Thus, the allowed energy states increase in energy while the Fermi energy stays the same, resulting in jumps in the resistivity of the material whenever one of the energy levels goes above the Fermi energy.

Now we attempt to find a more concrete way of saying how much the resistivity jumps when each of the energy levels goes above the Fermi energy. This can be explained simply by taking

$$n = \nu d \quad (12)$$

where $d = eB/h$ is the degeneracy of each state and $\nu \in \mathbb{N}$ is the label of a given energy level. More precisely, d gives the density of the number of electronic orbits allowed in the sample. Plugging d in we get that

$$B_\nu = \frac{nh}{\nu e}. \quad (13)$$

Note that only certain levels of the magnetic field cause changes as predicted by the quantum mechanics above. Finally, we simply plug in these values for the magnetic field into the Hall resistance given in (5) to obtain

$$R_H = \frac{h}{\nu e^2}. \quad (14)$$

III. METHODS

The sample was put in a circuit like Fig. 3 in which measured the voltage across the sample and the voltage along it using lockin amplifiers. A .1 volt AC signal at

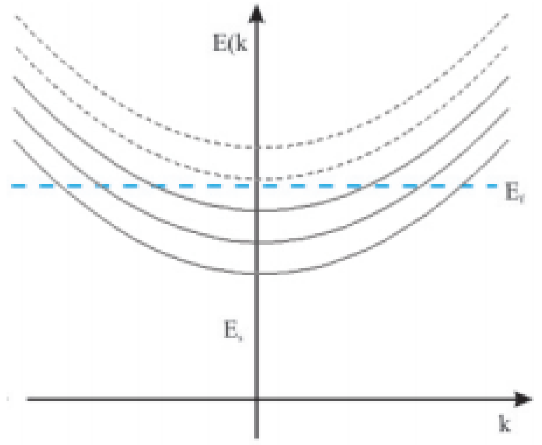

Figure 2: A qualitative diagram of the levels where electrons may sit in the material. As the magnetic field increases in intensity, the fermi-energy E_f will decrease and the number of levels that the charger carriers may occupy goes down. Thus increasing the resistance. K is the momentum of the particle. Diagram courtesy of [1].

100 Hz was sent across the sample in order to decrease the noise in our reading. This will be referred to as V_{drive} or the driving voltage.

An AsGe semiconductor was used to approximate a sheet of electrons like in Fig. 1. Semiconductors work well here because they often have large band gaps which hold electrons in place in a sheet, thus effectively removing the electron's freedom in one direction.

In order to get a reading of the resistance of the sample, a large resistance was put in parallel with the sample so that the current of the system was approximately constant overall. Thus, when the voltage was measured along the sample, Ohm's Law was applied to give the resistance. The resistance used $R = 9.8$ MΩ is more than enough to make the resistance change of the material negligible (on the order of 10^3) [3].

The voltages across and along the AsGe crystal were measured using lockin amplifiers with a time constant of 1ms. The sample was placed in a cooling chamber and first cooled with liquid nitrogen to 77K and then cooled to 4K using liquid helium. After that, the sample was cooled to 1K by reducing the pressure inside the container using a pumping system. All results discussed will be assumed to be taken at 1K unless otherwise specified.

IV. RESULTS

The results contained clear plateaus in the Hall voltage and clear peaks in the data taken along the sample (the longitudinal voltage) as can be clearly seen in Fig. 4. The plateaus were averaged together along the colorations shown in Fig. 4. It is clear to see that as the data goes lower and lower the plateaus are less and less discernible making the possibility of mis-labeling a plateau larger and larger.

It should be noted that at 4K, anomalous troughs were found after every plateau in the Hall voltage. However, these will be omitted from analysis as they disappeared once the sample was cooled to 1K.

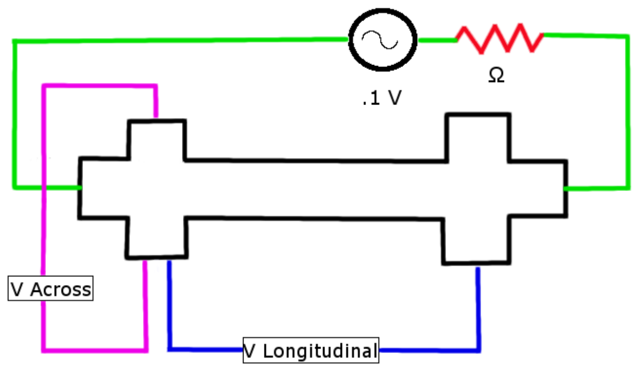

Figure 3: A diagram of the sample and where data was taken. An AC signal (the driving voltage) is sent across the AsGe semiconductor and across a resistor. The difference in voltages is measured across and along the sample.

Plateaus Measured in the Hall Voltage

1	$7.29\pm .02 \times 10^{-2}$ mV
2	$4.15 \pm .02 \times 10^{-2}$ mV
3	$2.92 \pm .02 \times 10^{-2}$ mV
4	$1.91 \pm .01 \times 10^{-2}$ mV
5	$1.28 \pm .01 \times 10^{-2}$ mV

Table I: A table of the plateaus of the Hall Voltages found in Fig. 4 with the rightmost plateau as 1 and the leftmost plateau as 5. Note that the voltages are very small for what they should be with a current with a magnitude of 10^{-5}.

V. ANALYSIS

A. Initial Calculations

From II we expect the plateaus in the resistance of the longitudinal signal to be multiples of e^2/h. To begin our quantitative analysis, we first must find the current across the sample. This can easily be done by assuming that the resistance of the sample is negligible (see III) and yields

$$I = \frac{V_{drive}}{R} = \frac{.1}{9.8191 \times 10^6} = 1.0204 \times 10^{-8} A \qquad (15)$$

where error has been neglected as both of these quantities have errors which are much less than that of the voltages in table I.

Analysis of the levels

Levels From Highest to Lowest	Measured Hall Resistance	Closest Levels (ν)
1	7.14 ± .02 ×10^3 mV	3.61 ± .01
2	4.07 ± .02 ×10^3 mV	6.34 ± .03
3	2.86 ± .01 ×10^3 mV	9.02 ± .06
4	1.87 ± .01 ×10^3 mV	13.8 ± .07
5	1.25 ± .01 ×10^3 mV	20.5 ± .58

Table II: A table of values calculated using table I and equations that can be found in II. Note the lack of correspondence of the calculated levels with any sort of pattern. The closest levels are the levels calculated using the known value for e^2/h (25818 Ω). If one wishes to calculate the Hall resistances with the resistance of the sample added in, all one must do is add $.02 \times 10^3$ to the first Hall resistance and $.01 \times 10^3$ to the rest. See V B for details on calculation.

Next we simply divide all the values in table I by the current using (5) to obtain table II. The level values (ν) were calculated simply by dividing R_H by the known value for e^2/h. It is clear to see from table I that there was little regularity in the levels.

We know from previous experiments that the top level is likely to be the 2nd level and the rest proceed as multiples of 2(a remark from Dr. Jesse Berezovsky). Using this model we can calculate e^2/h for each level and find the mean. If this analysis is performed we get that

$$\frac{e^2}{h} = 15050 \pm 38\Omega. \quad (16)$$

This estimate is very off from the actual result of 25818Ω and has an error of

$$\epsilon_{tot} = \frac{|\frac{e^2}{h} - result|}{\frac{e^2}{h}} = 41\%. \quad (17)$$

However, it is possible that the last 2 plateaus were misjudged as they are quite small. If we remove them from the data we find that

$$\frac{e^2}{h} = 15908 \pm 48\Omega. \quad (18)$$

and

$$\epsilon_0 = 38\%. \quad (19)$$

B. Attempted Correction

The reader may note that the values in'16 and 18 have a very low standard deviation, implying that all of the values averaged were about the same. This may be indicative of a missed scaling factor on the Hall voltage.

Taking the value given in 16, we find that the scale factor is roughly 8.56 ± .02.

Another correction that was attempted was assuming that the resistance of the sample was, in fact, not negligible. In order to solve for this new system we use Ohm's law manifest as

$$V_{drive} = I(R + R_s) = .1V = I(9.8 \times 10^6\Omega + R_s) \quad (20)$$

and

$$V_{Longitudinal} = IR_s \quad (21)$$

where R_s is the resistance of the sample, $V_{Longitudinal}$ is the voltage given in Fig. 4, and I is the current moving through the circuit. It is clear to see that for each value of $V_{Longitudinal}$ the system is solvable for I and R_s. When this was calculated, only small changes occurred in the 3rd significant figure of V_{Hall}, proving that the resistance of the sample was in fact negligible in the overall circuit.

VI. CONCLUSION

It is clear to see that the results obtained do not agree in any way with other data. Errors in the estimation of e^2/h differed by around 40% and the estimates for the levels were not even close to expected integers. Although this data is very hard to explain, it may be the product of a missed constant to multiply V_{Hall}. We can see evidence for this in (16) and (18) where the errors become very low despite the fact that the measured value is entirely different from the known value. This means that despite the fact that the values were all off, they were off by a similar relative scale factor since they were all less than the known value. It is hard to say where this scale factor may have come from. It was noted that our measured values for V_{Hall} were quite small so perhaps there was an error in reading the lockin or perhaps an incorrect resistance. However, the presence of plateaus in our raw data suggests that our basic procedure was correct.

The longitudinal voltage behaved normally giving clear peaks right at the plateaus of the Hall voltage. However, the longitudinal voltage appears to rise linearly with the magnetic field rather than being centered around 0. This could be caused by several factors but the most likely is some of the Hall voltage being measured in the longitudinal voltage. This mixing of voltages is likely because in an experiment taken at 4K, it was observed that the longitudinal voltage was being mixed with the Hall Voltage. This could be due to an error wiring that may have occurred when the sample was removed.

Despite the lack of agreement with known results, we were able to eliminate several sources of error like the resistance of the sample and were able to achieve order to magnitude correctness in our data.

VII. SUGGESTIONS FOR FURTHER RESEARCH

Perhaps in the future groups ought to review the lockin amplifier to make sure that it is working properly before use. Also, a list of items required for the experiment would be helpful as there was confusion as to the role of the different parts in the beginning of the experiment.

Acknowledgments

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[2] D.B. Pengra, J. Stoltenberg, R. Van Dyck, and O. Vilches,The Hall Effect, Courses.Washington.Edu (2015).
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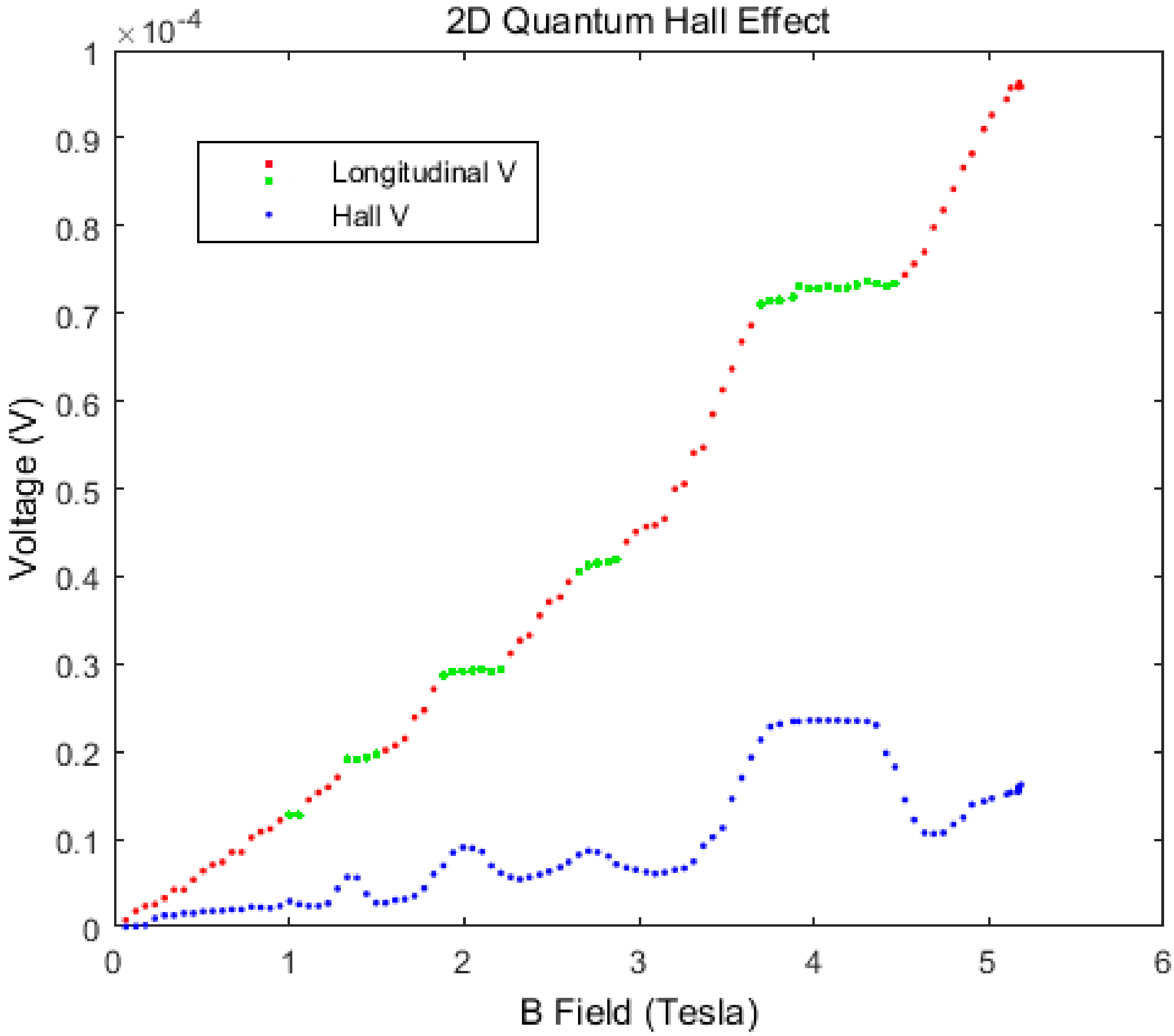

Figure 4: Raw data obtained for the voltages across the sample and along the sample as a function of the applied magnetic field (see Fig. 1 for details). It is clear to see the expected plateaus in the Hall voltage as a result of the 2-dimensional quantum Hall effect. The green points are the ones that were averaged to get the results seen in table I Note that this graph is off by a scaling factor as the effect of the gain due to the lockin has not been taken into account.